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Keywords

PROPERTIES OF CERTAIN CLASSES OF ANALYTIC OR HARMONIC FUNCTIONS

PRESA UNIVERSITARĂ CLUJEANĂ

2019

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Chapter 1

Preliminary results

1.1 Introduction

Complex analysis is a branch of mathematics that has various scientific and technical applications.

From the beginning of the XX-th century both real and complex analysis have gone through major development due to natural internal progress or to satisfy the needs of different branches of mathematics or even other sciences.

The basics of geometric theory of functions of one complex variable, with roots in the XIX-th century or even earlier, were set by the work of P. Koebe (1907) and L. Bieberbach, whose famous conjecture from 1916 was proved only in 1984 by Louis de Branges. The analytic functions of one complex variable are the ideal model for geometric transformations of the plane.

One of the most important centers in the field of geometric theory of functions was in Cluj, where G. Călugăreanu obtained some important results in 1931, establishing the first necessary and sufficient conditions of univalence expressed by coefficients.

One of the fields that aroused the interest of a large number of mathematicians all over the world was that of the geometric theory of functions of one or several complex variables, a special branch of complex analysis. In this field there are some very interesting problems to study as those referring at differential subordinations, integral or differential operators for some classes of functions, other properties of some classes of analytic, univalent or multivalent functions, some of them with negative coefficients.

The school of complex analysis from Cluj, under the leadership of professor Petru T. Mocanu contributed with some important results. Mocanu, together with S. S. Miller, introduced the method of differential subordination also known as method of admissible function. Their method was further developed by other mathematicians who demonstrated in a much simpler way some classical results in this area, adding expansions, and even new results to the study of several integral operators, conditions for starlikeness and convexity, preserving of some geometric and analytic properties of several integral and differential operators. At this moment the theory of differential subordinations is studied and successfully used by mathematicians all over the world, from U.S.A to Germany, Poland, Turkey, India, China, Japan, Canada, Malaysia or Egypt.

Miller and Mocanu introduced also the notion of differential superordination, dual notion of the differential subordination. The theory of subordination chains is one of the most modern theories of complex analysis that give new research directions for some important problems related to univalent functions.

Another recent research method is based on the use of some conditions of coefficients developed in power series of analytic functions (with positive and negative coefficients). Using this method some well appreciated results were obtained, cited by mathematicians all over the world. Nowadays the complex

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analysis is well represented in Cluj through the activity of scientists as G. Ș. Sălăgean, G. Kohr or T. Bulboacă, as well as in other university centers of Romania.

The present book contains five chapters.

In **Chapter 1** we present the historical background of the geometric function theory and we give some basic notations and preliminary results.

Chapter 2 (Analytic functions with varying arguments) contains four subchapters. In this chapter we study some properties of certain class of analytic functions with varying arguments defined by Ruscheweyh and Sălăgean derivative and the combination of these derivatives.

Chapter 3 (Analytic functions) This chapter is dedicated to the study of coefficient bounds and Fekete-Szegő problem for new classes of analytic functions with varying arguments defined by Sălăgean integro-differential operator.

Chapter 4 (Differential subordinations and superordinations) contains six subchapters.

In the first subchapter we consider the $\mathcal{D}I^n : \mathcal{A} \rightarrow \mathcal{A}$, $\mathcal{D}I^n f(z) = (1 - \lambda) \mathcal{D}^n f(z) + \lambda I^n f(z)$ linear operator, where $\mathcal{D}^n$ is the Sălăgean differential operator and I^n is the Sălăgean integral operator. We study several differential subordinations generated by $\mathcal{D}I^n$ and we introduce a class of holomorphic functions $L_n^m(\beta)$, and obtain some subordination results.

In the second subchapter we define the operator $\mathcal{D}_{\alpha,\beta}^{\lambda,v,n} : \mathcal{A} \rightarrow \mathcal{A}$, given by $\mathcal{D}_{\alpha,\beta}^{\lambda,v,n} f(z) = (1 - \alpha - \beta) \mathcal{R}^v \mathcal{D}^n f(z) + \alpha \mathcal{R}^v \Omega_z^\lambda f(z) + \beta \mathcal{D}^n \Omega_z^\lambda f(z)$, for $z \in U$, where $\mathcal{R}^v$ is the Ruscheweyh derivative, $\mathcal{D}^n$ is the Sălăgean operator, Ω_z^λ is a fractional differintegral operator introduced by S. Owa and H. M. Srivastava, $\mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + a_3 z^3 + \dots, z \in U\}$, $\alpha, \beta \geq 0, v > -1$, $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, $-\infty < \lambda < 2$. A certain subclass of analytic functions in

the open unit disk, $\mathcal{R}_{\alpha,\beta}^{\lambda,v,n}(\delta)$, where $0 \leq \delta \leq 1$, is introduced using the new operator. We obtain some properties of the class $\mathcal{R}_{\alpha,\beta}^{\lambda,v,n}(\delta)$ and some differential subordinations using the operator $\mathcal{D}_{\alpha,\beta}^{\lambda,v,n}$.

In the third subchapter we use the $\mathcal{DI}^n$ operator defined in the first subchapter. We give some results and applications for differential subordinations and superordinations for analytic functions and we will determine some properties on admissible functions defined with the new operator.

The next part is dedicated to the presentation of Loewner chains and their utility in obtaining new univalence criteria.

In the fifth subchapter we study the properties of the image of some subclasses of starlike functions, through the generalized Bernardi - Libera - Livingston integral operator. A new subclass of functions with negative coefficients is introduced and we study some properties of this class.

In the sixth subchapter we determine the radius of convexity of particular functions. The obtained results will be used to deduce sharp estimations regarding functions which satisfy a second order differential subordination. A lemma regarding starlikeness is deduced which involves the notion of convolution. This lemma is used in order to obtain a sharp starlikeness condition.

In **Chapter 5** (Harmonic functions), with two subchapters, we investigate several classes of harmonic functions with varying argument of coefficients. Properties as the coefficient estimates, distortion theorems, convolution properties, radii of convexity, starlikeness and the closure properties of these classes under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_c(f)$, ($c > -1$) which is defined by $\mathcal{L}_c(f) = \mathcal{L}_c(h) + \overline{\mathcal{L}_c(g)}$ where

$$\mathcal{L}_c(h)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) dt \quad \text{and} \quad \mathcal{L}_c(g)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} g(t) dt$$

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are investigated.

1.2 Definitions and notations

Let $\mathbb{R}$ be the field of real numbers. Also let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ while $\mathbb{Z}$ is the set of integers.

Let $\mathbb{C}$ be the complex plane, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the extended complex plane.

Let $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ be the open disk of radius $r > 0$, centered at $z_0 \in \mathbb{C}$.

The closure of $U(z_0, r)$ will be denoted by $\bar{U}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ and its boundary by $\partial U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$. Let $\dot{U}(z_0, r) = U(z_0, r) \setminus \{z_0\}$.

The disk $U(0, r)$ is denoted by U_r and the unit disk is denoted by $U_1 = U = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1.1. [26] A complex-valued function $f : G \rightarrow \mathbb{C}$ (G open set) of a complex variable is differentiable at a point $z_0 \in G$ if it possesses the complex derivative at z_0 :

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}.$$

Definition 1.2. A function is analytic if and only if its Taylor series expansion about z_0 converges to the function in some neighborhood for every z_0 in its domain:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \text{ where } a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Definition 1.3. A complex-valued function $f : G \rightarrow \mathbb{C}$ (G open set) of a complex variable is called a holomorphic function at z_0 if it is differentiable at every point in some neighborhood of z_0 . Let $G \subseteq \mathbb{C}$ be an open set. We denote by $H(G)$ the set of holomorphic functions defined on G with values

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in $\mathbb{C}$. Holomorphic functions on the whole complex plane are called entire functions.

Let $H(U)$ be the set of holomorphic functions in U .

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + \dots\}$$

and

$$\mathcal{A}_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \dots\}, \text{ with } \mathcal{A} = \mathcal{A}_1.$$

The Taylor series expansion of a function $f \in \mathcal{A}$ is :

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

1.3 Univalent functions

Definition 1.4. [26] A single-valued function $f \in H(D)$ is said to be univalent (or schlicht) in a domain $D \subset \mathbb{C}$ if f is injective (it never takes the same value twice). We denote by $H_u(D)$ the class of univalent functions in D .

Theorem 1.1. [39] If $f \in H_u(D)$ then $f'(z) \neq 0, \forall z \in D$.

Corollary 1.1. [56],[36] (Noshiro-Warschawski-Wolff theorem)

If D is a convex domain and $f \in H(D)$ such that $\Re f'(z) > 0$, for any $z \in D$, then $f \in H_u(D)$.

We denote by $S = \{f \in \mathcal{A} : f \in H_u(U)\}$ the class of univalent function in the unit disk U normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Denote with S^* the class of starlike functions in U :

$$(1.2) \quad S^* = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > 0, z \in U \right\}, S^* \subset S.$$

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Denote with K the class of convex functions in U :

$$(1.3) \quad K = \left\{ f \in \mathcal{A} : \Re \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}, K \subset S^* \subset S$$

Definition 1.5. Let r_f be the radius of convergence of the function f . The radius of convexity of the function f is

$$(1.4) \quad r_f^c = \sup \left\{ r \in (0, r_f) \mid \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \forall z \in U_r \right\}.$$

Definition 1.6. ([56], *def.* 3.5.1) Let f and g be analytic functions in U . We say that the function f is subordinate to the function g , if there exists a function w , which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z)); \forall z \in U$. We denote by $<$ the subordination relation. If g is univalent, then $f < g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

1.4 Differential and integral operators

Definition 1.7. [94]

For $f \in \mathcal{A}$, $n \in \mathbb{N}_0$, the Sălăgean differential operator $\mathcal{D}^n$ is defined by $\mathcal{D}^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\mathcal{D}^0 f(z) = f(z),$$

$$\mathcal{D}^1 f(z) = zf'(z),$$

$$\mathcal{D}^{n+1} f(z) = z(\mathcal{D}^n f(z))', z \in U$$

Remark 1.1. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$(1.5) \quad \mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.$$

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Definition 1.8. [94]

For $f \in \mathcal{A}$, $n \in \mathbb{N}_0$, the Sălăgean integral operator I^n is defined by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt, \dots$$

$$I^{n+1} f(z) = I(I^n f(z)), z \in U$$

I^1 is the Alexander operator used for the first time in [7] and I^n operator is called the generalized Alexander operator.

Remark 1.2. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$(1.6) \quad I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k,$$

$z \in U$, $n \in \mathbb{N}_0$ and $z(I^{n+1} f(z))' = I^n f(z)$.

Remark 1.3. We have $\mathcal{D}^n I^n f(z) = I^n \mathcal{D}^n f(z) = f(z)$, $f \in \mathcal{A}$, $z \in U$.

Definition 1.9. Let $\lambda \geq 0$, $n \in \mathbb{N}_0$. Denote by $\mathcal{D}I^n$ the operator given by $\mathcal{D}I^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\mathcal{D}I^n f(z) = (1 - \lambda) \mathcal{D}^n f(z) + \lambda I^n f(z), z \in U.$$

Remark 1.4. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$(1.7) \quad \mathcal{D}I^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1 - \lambda) + \lambda \frac{1}{k^n} \right] a_k z^k, z \in U.$$

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Definition 1.10. Let $f, g \in H(U)$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. The convolution (or Hadamard product) of the functions f and g is defined by

$$(1.8) \quad (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

The modified Hadamard product is ([40], [92], [95])

$$(1.9) \quad (f \otimes g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g \otimes f)(z).$$

Definition 1.11. Ruscheweyh [90] defined the derivative $\mathcal{R}^\gamma : \mathcal{A} \rightarrow \mathcal{A}$ by

$$(1.10) \quad \mathcal{R}^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), (\gamma \geq -1), f \in \mathcal{A}, z \in U.$$

In the particular case $n \in \mathbb{N}_0$

$$(1.11) \quad \mathcal{R}^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}.$$

The symbol $\mathcal{R}^n f(z)$ ($n \in \mathbb{N}_0$) was called the n -th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [2]. It is easy to see that

$$\mathcal{R}^0 f(z) = f(z),$$

$$\mathcal{R}^1 f(z) = z f'(z), \dots$$

$$(n+1)\mathcal{R}^{n+1} f(z) = z(\mathcal{R}^n f(z))' + n\mathcal{R}^n f(z), z \in U.$$

Remark 1.5. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$(1.12) \quad \mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} a_k z^k = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)} a_k z^k, n > -1, z \in U,$$

or

$$(1.13) \quad \mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \delta(n, k) a_k z^k, \text{ where } \delta(n, k) = \binom{n+k-1}{n} z \in U,$$

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Definition 1.12. [5] Let $\lambda \geq 0, n \in \mathbb{N}_0$. Denote by $\mathcal{R}\mathcal{D}^n$ the operator given by $\mathcal{R}\mathcal{D}^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\mathcal{R}\mathcal{D}^n f(z) = (1 - \lambda) \mathcal{R}^n f(z) + \lambda \mathcal{D}^n f(z), z \in U.$$

Remark 1.6. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$(1.14) \quad \mathcal{R}\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} \left\{ (1 - \lambda) \frac{(n + k - 1)!}{n! (k - 1)!} + \lambda k^n \right\} a_k z^k, z \in U.$$

Definition 1.13. [4]

For $f \in \mathcal{A}, \lambda \geq 0$ and $n \in \mathbb{N}_0$, the operator $\mathcal{D}_\lambda^n$ is defined by $\mathcal{D}_\lambda^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\mathcal{D}_\lambda^0 f(z) = f(z),$$

$$\mathcal{D}_\lambda^1 f(z) = (1 - \lambda) f(z) + \lambda z f'(z) = \mathcal{D}_\lambda f(z), \dots$$

$$\mathcal{D}_\lambda^{n+1} f(z) = (1 - \lambda) \mathcal{D}_\lambda^n f(z) + \lambda z (\mathcal{D}_\lambda^n f(z))' = \mathcal{D}_\lambda (\mathcal{D}_\lambda^n f(z)), z \in U$$

Remark 1.7. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$(1.15) \quad \mathcal{D}_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k - 1) \lambda]^n a_k z^k, z \in U.$$

Remark 1.8. For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [94].

Definition 1.14. Let $\gamma, \lambda \geq 0, n \in \mathbb{N}_0$. Denote by $\mathcal{R}\mathcal{D}_\lambda^n$ the operator given by $\mathcal{R}\mathcal{D}_\lambda^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\mathcal{R}\mathcal{D}_\lambda^n f(z) = (1 - \gamma) \mathcal{R}^n f(z) + \gamma \mathcal{D}_\lambda^n f(z), z \in U.$$